Newton's method

1. Use three steps of Newton's method to approximate a root of the function $f(x) \overset{\text{def}}{=} \frac{\sin(x)}{x} + e^{-x}$ starting with $x_0 = 3.0$.

Answer: To ten significant digits, we have 3.0, 3.244843859, 3.266322922, 3.266500425

2. Given that the absolute error at one step of Newton's method is $|x_k - r|$ where r is a root of a function f, demonstrate that the absolute error $|x_{k+1} - r|$ is proportional to a value multiplied by $|x_k - r|^2$ under the assumption that $1 < |f^{(1)}(x)| < 2$ in the vicinity of the root, and $|f^{(2)}(x)| < 5$ in the vicinity of the root.

Answer: See the course notes.

3. Use three steps of Newton's method to approximate a root of the function $f(x) \overset{\text{def}}{=} x^3 - 3x + 1$ starting with $x_0 = -0.6776507$

Answer: To ten significant digits, we have –0.6776507, 1.000000005, 33145927.27, 22097284.84.

4. What is the cause for the sequence of approximations in Question 3?

Answer: The function has a derivative equal to zero when $x = 1$, so when x_1 is approximately equal to one, the derivative is very large.

5. If you continue to iterate Newton's method in Question 3, what root does it converge to?

Answer: To ten significant digits, 1.532088886

6. If you iterated Newton's method in Question 3 but starting with $x_0 = -0.6$, what root does it converge to?

Answer: To ten significant digits, 0.3472963553.

7. In general, should you apply Newton's method if you don't already have an idea as to what a root of a function is?

Answer: In general, no. Newton's method is a tool to refine an approximation of a root, not to check if a function has a root. If you start with an arbitrary initial point, it may or may not converge to a root if there is one, so non-convergence does not suggest there is no root.

8. The function x^2 has a double root at $x = 0$. Apply Newton's method starting with the initial value $x_0 = 1$. Is the convergence still $O(h^2)$?

Answer: No, with a double root, the rate of convergence becomes $O(h)$.